

Wytrzymałość konstrukcji 1

Wykład 7

Charakterystyki geometryczne figur płaskich

Wskaźniki charakteryzujące przekrój pręta (figurę płaską)

A – pole

Momenty statyczne:

$$S_{\eta} = \int_A \zeta dA$$

$$S_{\zeta} = \int_A \eta dA$$

Momenty bezwładności:

$$J_{\eta} = \int_A \zeta^2 dA$$

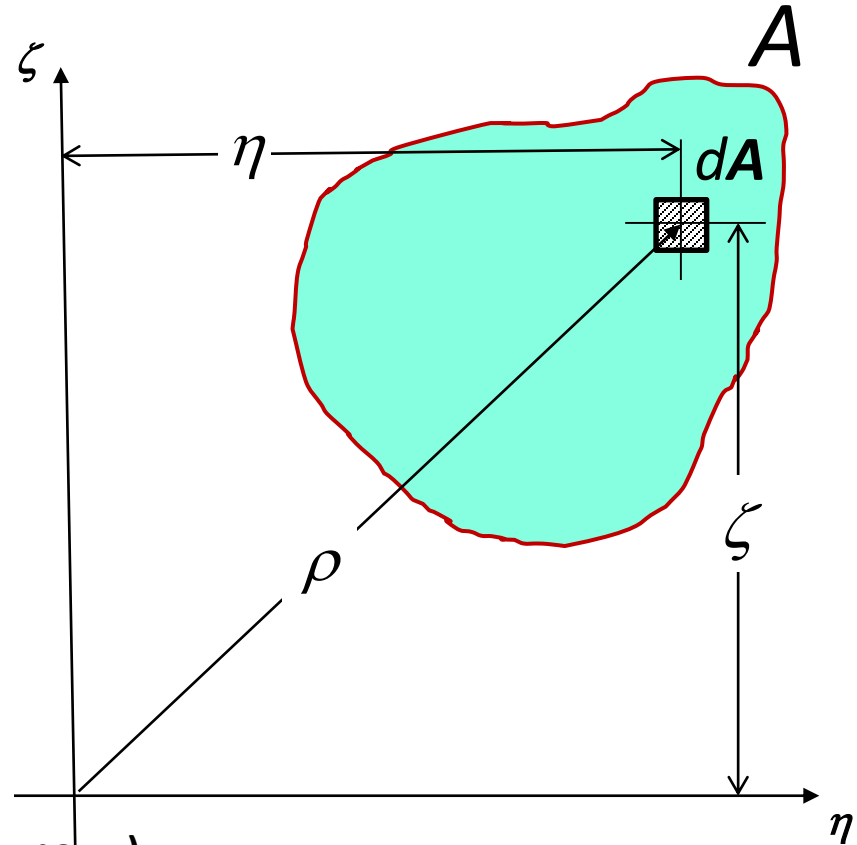
$$J_{\zeta} = \int_A \eta^2 dA$$

Moment dewiacji (odśrodkowy):

$$J_{\eta\zeta} = \int_A \eta \zeta dA$$

Momenty bezwładności względem punktu (biegunowy):

$$J_0 = \int_A \rho^2 dA = J_{\eta} + J_{\zeta}$$



Centralne osie bezwładności

(y, z) to osie *centralne* (przechodzą przez środek ciężkości figury) i równoległe do osi układu (η, ζ)

Wtedy momenty statyczne:

$$S_y = \int_A z dA = 0$$

$$S_z = \int_A y dA = 0$$

Wtedy współrzędne środka ciężkości:

$$S_\eta = \int_A \zeta dA = \zeta_c A \Rightarrow \zeta_c = \frac{1}{A} \int_A \zeta dA$$

$$S_\zeta = \int_A \eta dA = \eta_c A \Rightarrow \eta_c = \frac{1}{A} \int_A \eta dA$$

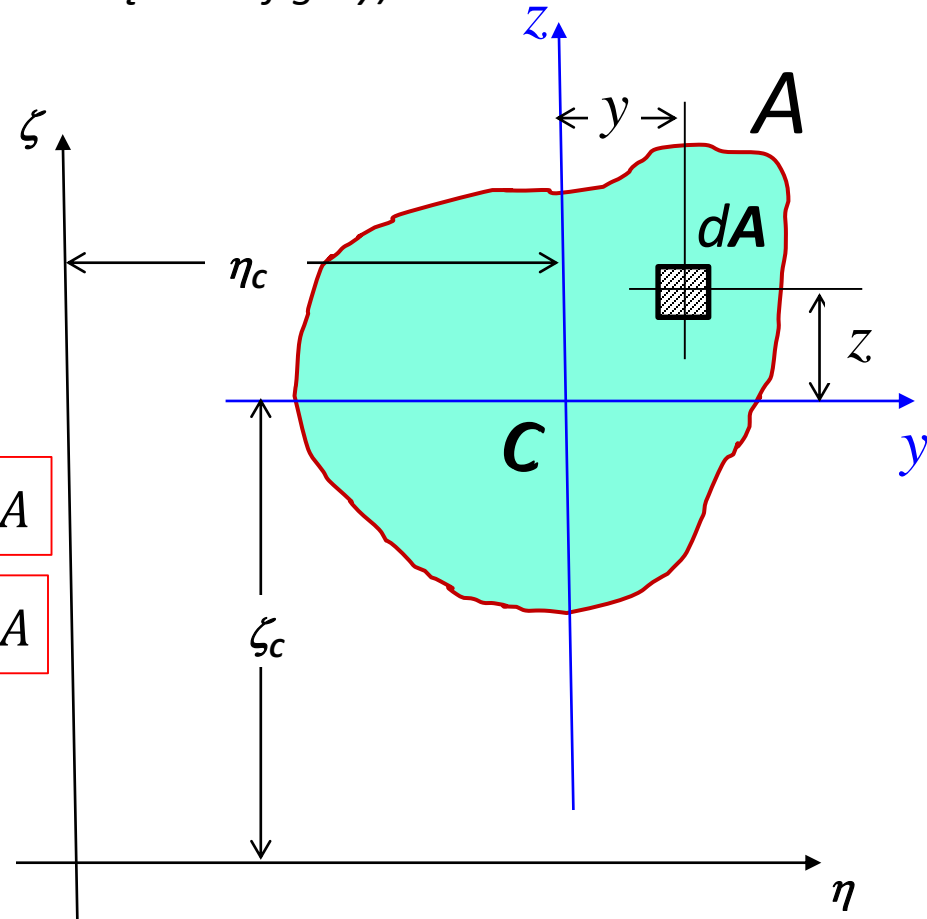
Centralne momenty bezwładności:

$$J_y = \int_A z^2 dA$$

$$J_z = \int_A y^2 dA$$

Centralny moment dewiacji (odśrodkowy):

$$J_{yz} = \int_A y z dA$$



Twierdzenie Steinera

Jeśli (y, z) to osie *centralne* i równoległe do osi układu (η, ζ)

$$J_{\eta} = J_y + A \zeta_c^2$$

$$J_{\zeta} = J_z + A \eta_c^2$$

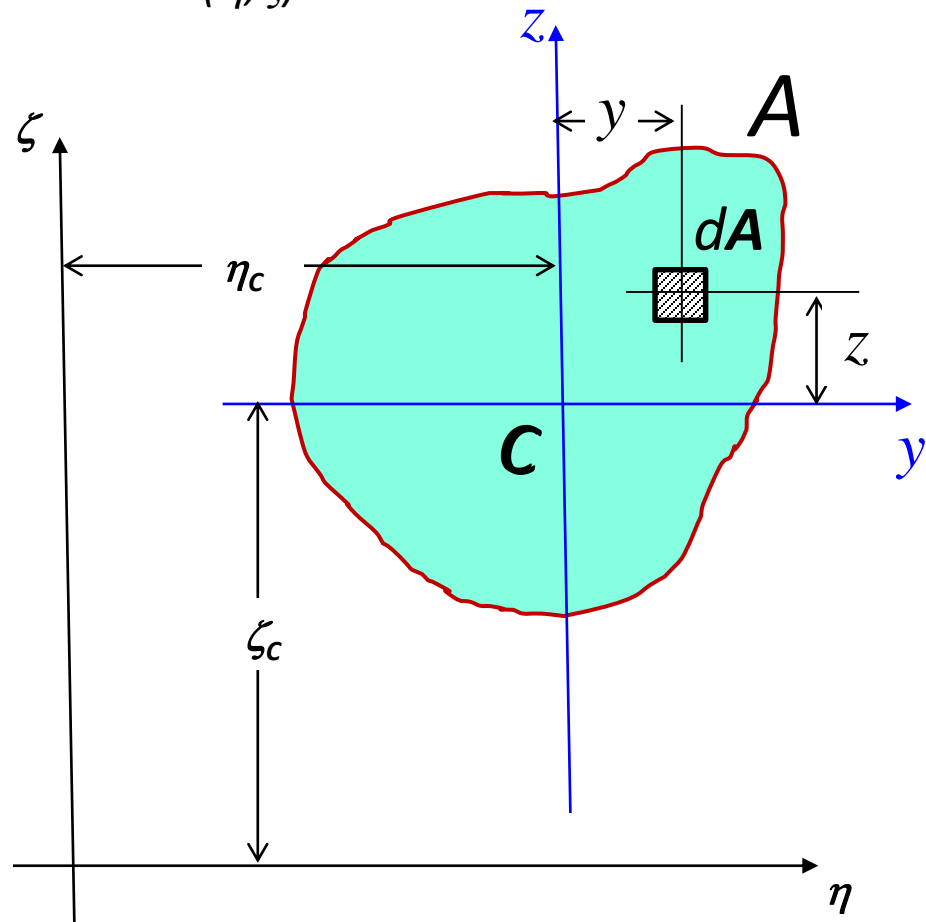
$$J_{\eta\zeta} = J_{yz} + A \eta_c \zeta_c$$

Jeśli:

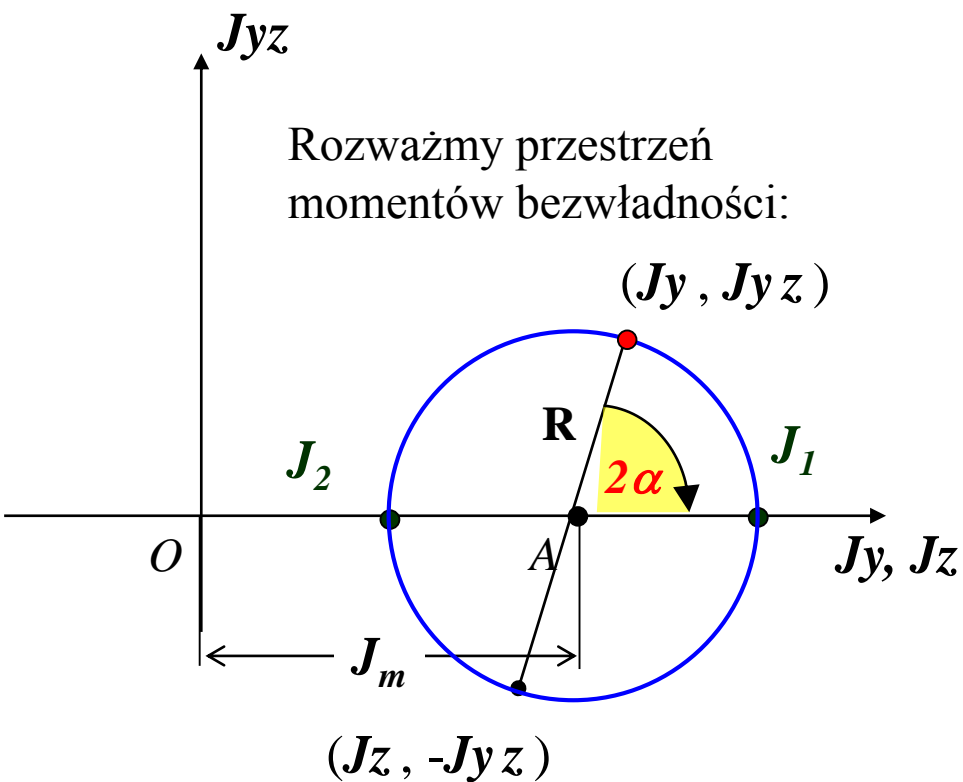
$$J_{\eta\zeta} = 0 \quad \rightarrow \text{osie główne}$$

$$S_y = 0 ; S_z = 0 \quad \rightarrow \text{osie centralne}$$

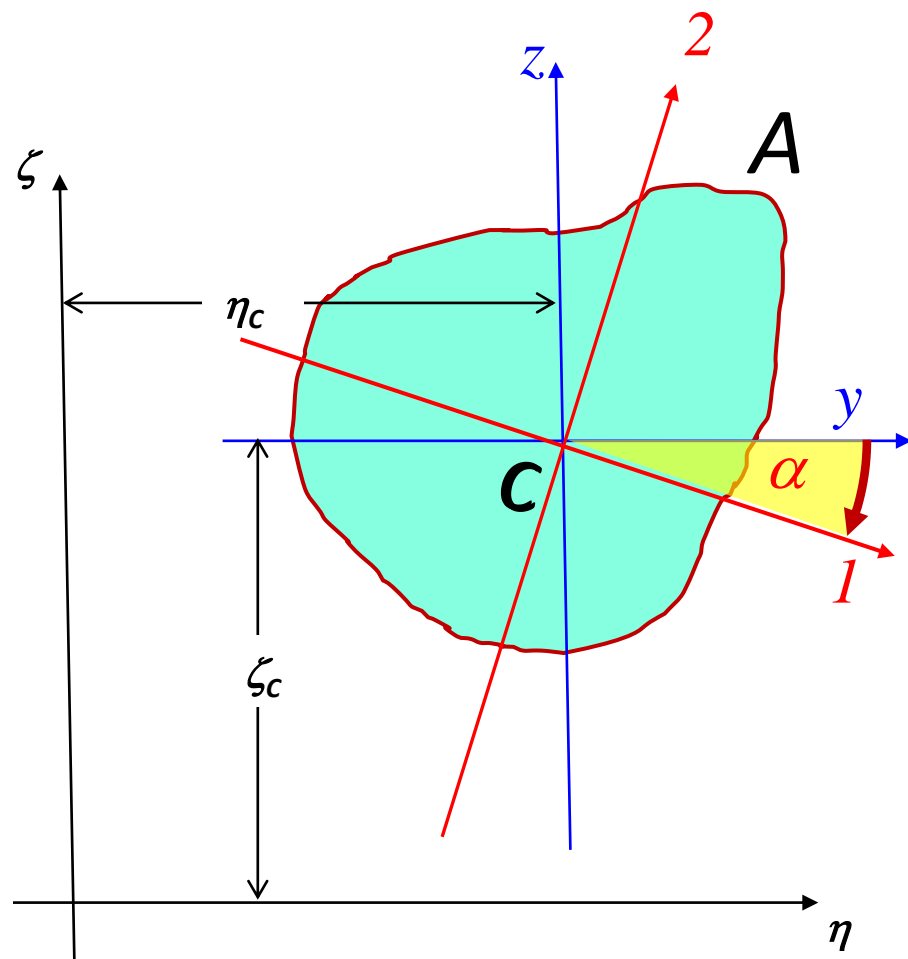
$$J_{yz} = 0 \quad \rightarrow \text{Osie główne i centralne}$$



Wyznaczenie centralnych głównych momentów bezwładności

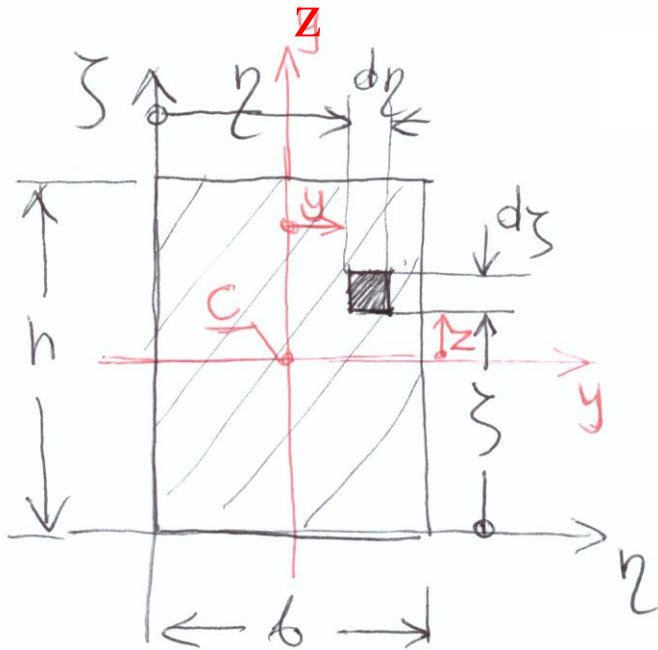


$(1, 2)$ to osie centralne główne



Zadanie 7.1

Zad. 1. Wyznaczyć centralne główne momenty bezwładności



Wyznaczymy S_C :

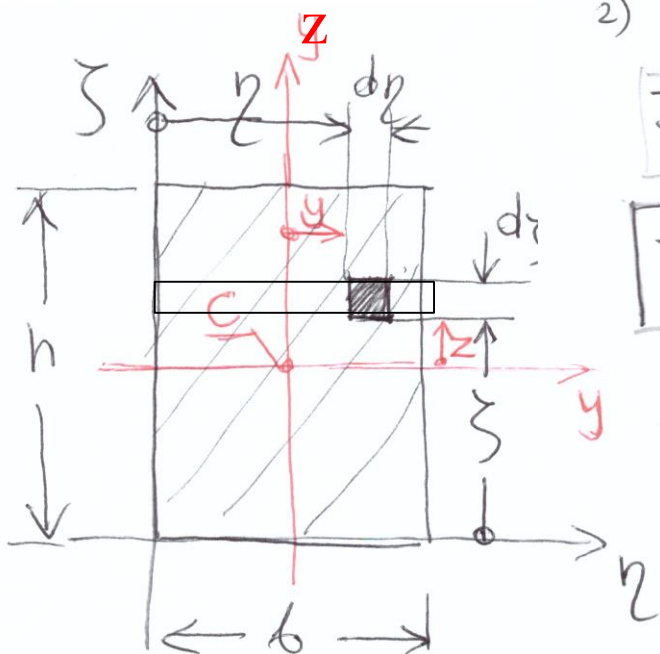
$$S_\eta = \int_A \zeta dA = \zeta_0 \cdot A$$

$$\zeta_0 = \frac{1}{A} \int_A \zeta dA = \frac{1}{b \cdot h} \int_0^h \int_0^b \zeta \cdot d\eta d\zeta$$

$$= \frac{1}{b \cdot h} \cdot b \int_0^h \zeta d\zeta = \frac{1}{h} \cdot \frac{1}{2} \zeta^2 \Big|_0^h = \underline{\underline{\frac{h}{2}}}$$

Podobnie $\eta_c = \frac{b}{2}$

Zadanie 7.1 (cd)



2) Momenty bezwładności względem osi centralnej

$$J_y = \int_A z^2 dA = \int_{-h/2}^{h/2} z^2 b dz = b \cdot \frac{1}{3} z^3 \Big|_{-h/2}^{h/2} = \frac{1}{3} b \left(\frac{h^3}{8} - \left(-\frac{h^3}{8}\right) \right)$$

$$J_y = \frac{bh^3}{12}$$

Podobnie $J_z = \frac{h \cdot b^3}{12}$

3) Moment dewiacji

$$J_{yz} = \int_A yz dA = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} yz dy dz = \int_{-h/2}^{h/2} z \left(\frac{1}{2} y^2 \Big|_{-b/2}^{b/2} \right) dz = 0$$

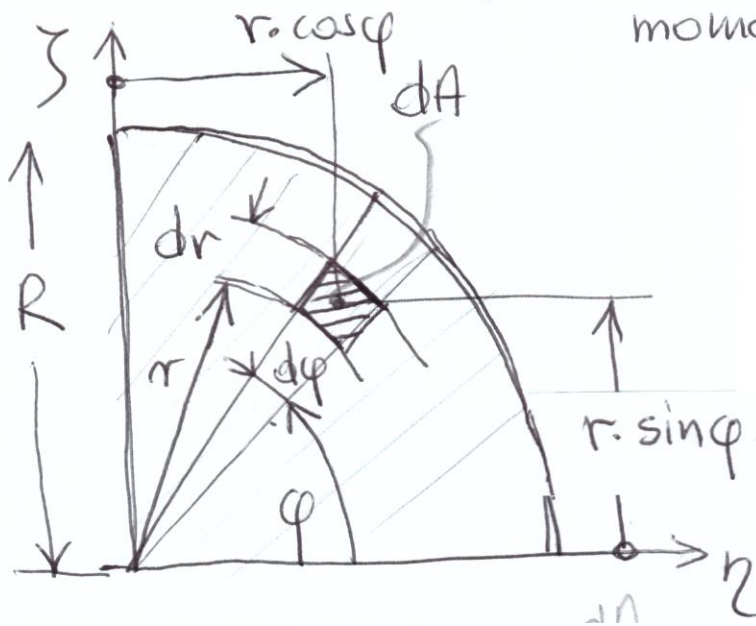
$$J_{yz} = 0$$

A więc osie główne i centralne

Wnioski: Osie symetrii są osiami głównymi

Zadanie 7.2

Zad. 2. Wyznaczyć centroidę, grawitacyjne momenty bezwładności



1) Pole $A = \frac{\pi R^2}{4}$

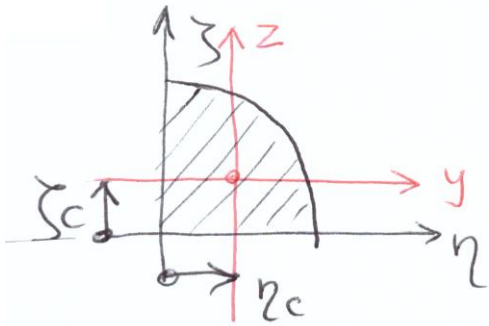
2) środek ciężkości

$$\zeta_c = \frac{1}{A} \int_A \zeta \cdot dA$$

$$\begin{aligned} \zeta_c &= \frac{1}{A} \int_0^R \int_0^{\pi/2} r \sin \varphi \, dr \, r d\varphi = \frac{1}{A} \int_0^R r^2 \int_0^{\pi/2} \sin \varphi \, d\varphi \, dr = \\ &= \frac{1}{A} \int_0^R r^2 (-\cos \varphi \Big|_0^{\pi/2}) \, dr = \frac{1}{A} \int_0^R r^2 \, dr = \frac{4}{3} \frac{R^3}{\pi R^2} \end{aligned}$$

$$\zeta_c = \frac{4}{3} \frac{R}{\pi} \quad \eta_c = \frac{4}{3} \frac{R}{\pi}$$

Zadanie 7.2 (c.d.)



3) Momenty bezwładności

$$\begin{aligned}
 J_{\eta} &= \int_A \zeta^2 dA = \int_0^R \int_0^{\pi/2} r^2 \sin^2 \varphi \cdot r d\varphi dr = \\
 &= \int_0^R r^3 \left(\int_0^{\pi/2} \sin^2 \varphi d\varphi \right) dr = \int_0^R r^3 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2} \cos 2\varphi \right) d\varphi dr \\
 &= \int_0^R r^3 \frac{1}{2} \left(\varphi \Big|_0^{\pi/2} - \frac{1}{2} \sin 2\varphi \Big|_0^{\pi/2} \right) dr = \frac{\pi}{4} \int_0^R r^3 dr = \frac{\pi}{16} R^4
 \end{aligned}$$

$$\boxed{J_{\eta} = \frac{\pi}{16} R^4} \quad \boxed{J_{\zeta} = \frac{\pi}{16} R^4}$$

4) Moment dewiągi

$$\begin{aligned}
 J_{\eta\zeta} &= \int_A \zeta \eta dA = \int_0^R \int_0^{\pi/2} r \sin \varphi \cdot r \cos \varphi \cdot r d\varphi dr = \\
 &= \int_0^R r^3 \int_0^{\pi/2} \frac{1}{2} \sin 2\varphi d\varphi dr = \frac{1}{4} \int_0^R r^3 \left(-\cos 2\varphi \Big|_0^{\pi/2} \right) dr
 \end{aligned}$$

$$\boxed{J_{\eta\zeta} = \frac{1}{2} \int_0^R r^3 dr = \frac{1}{8} R^4}$$

Zadanie 7.2 (c.d.)

5) Momenty centralne

Tw. Steinera: $J_{\eta} = J_y + A z_c^2$

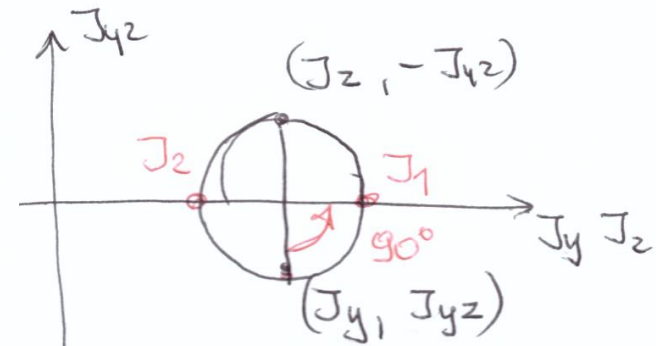
$$J_y = J_{\eta} - A z_c^2 = \frac{\pi}{16} R^4 - \frac{\pi R^2}{4} \left(\frac{4}{3} \frac{R}{\pi}\right)^2$$

$$J_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) R^4 \approx \boxed{0.05488 R^4} = J_z$$

$$J_{yz} = J_{\eta z} - A z_c \eta_c$$

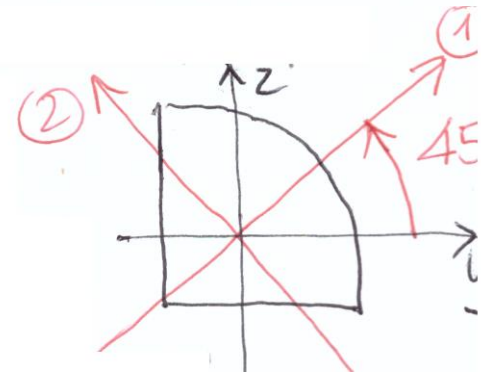
$$J_{yz} = \frac{1}{8} R^4 - \frac{\pi R^2}{4} \left(\frac{4}{3} \frac{R}{\pi}\right) \left(\frac{4}{3} \frac{R}{\pi}\right)$$

$$J_{yz} = \left(\frac{1}{8} - \frac{4}{9\pi}\right) R^4 \approx \boxed{-0.0165 R^4}$$

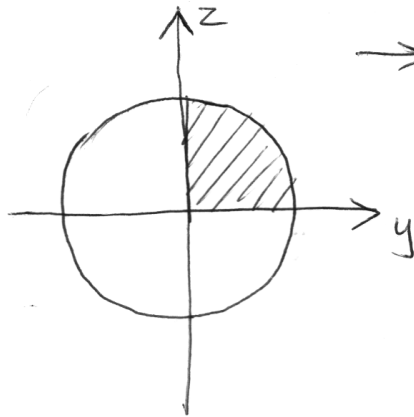


$$J_1 = 0.07138 R^4$$

$$J_2 = 0.03838 R^4$$



Popularne przekroje – warto znać na pamięć te wzory!

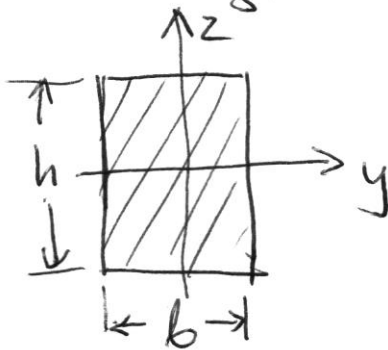


→ Pełne koło

$$J_y = 4 \times \frac{\pi}{16} R^4 = \frac{\pi}{4} R^4$$

$$J_y = \frac{\pi D^4}{64}$$

→ Prostokąt



$$J_y = \frac{bh^3}{12}$$

$$J_z = \frac{hb^3}{12}$$

Chwyty stosowane w obliczeniach

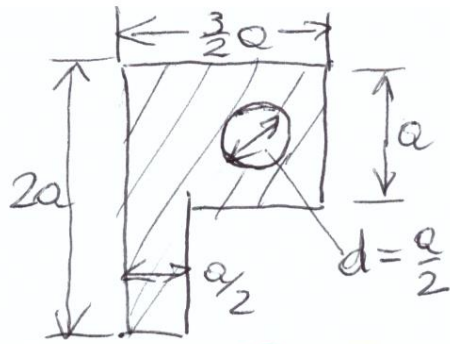
CHWYTY

$$J_y = \text{[Diagram of a ring with outer diameter } D_2 \text{ and inner diameter } D_w \text{]} = \text{[Diagram of a solid circle with diameter } D_2 \text{]} - \text{[Diagram of a solid circle with diameter } D_w \text{]} = \boxed{\frac{\pi}{64} (D_2^4 - D_w^4)}$$

$$J_y = \text{[Diagram of a solid rectangle with width } B \text{ and height } H \text{]} - \text{[Diagram of a solid rectangle with width } (B-\delta) \text{ and height } h \text{]} = \boxed{\frac{BH^3}{12} - \frac{(B-\delta)h^3}{12}}$$

$$J_z = \text{[Diagram of a flange]} + \text{[Diagram of a web]} = \boxed{\frac{(H-h)B^3}{12} + \frac{h\delta^3}{12}}$$

Zadanie 7.2



$$A = A_I + A_{II} - A_{III}$$



$$A_I = a^2$$

$$\eta_{cI} = -\frac{3}{4}a$$

$$\zeta_{cI} = -\frac{1}{2}a$$



$$A_{II} = a^2$$

$$\eta_{cII} = 0$$

$$\zeta_{cII} = 0$$



$$A_{III} = \frac{\pi a^2}{16}$$

$$\eta_{cIII} = 0$$

$$\zeta_{cIII} = 0$$

$$J_{ycI} = \frac{a}{2} \frac{(2a)^3}{12}$$

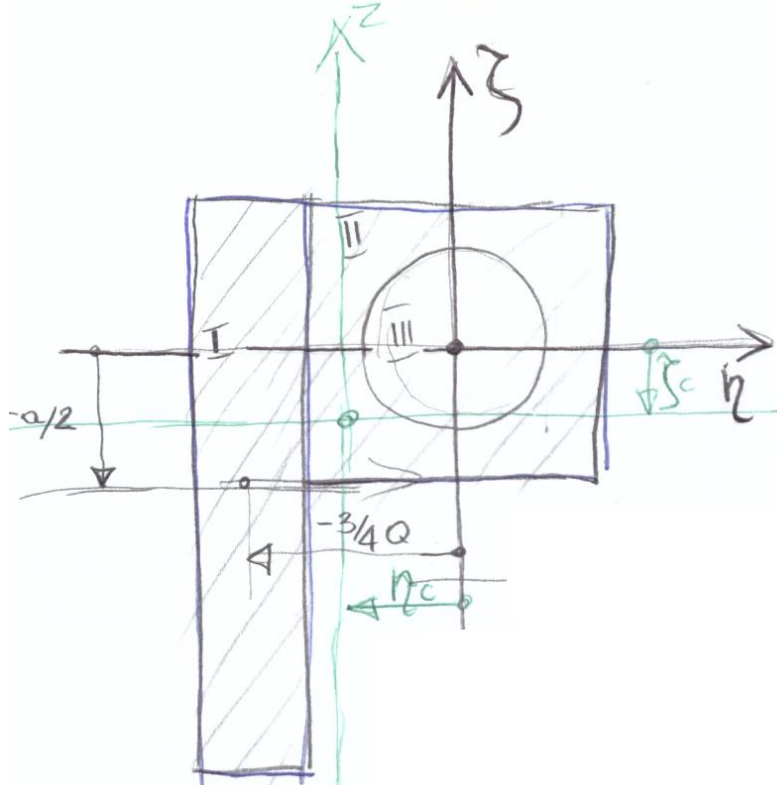
$$J_{ycII} = \frac{a^4}{12}$$

$$J_{ycIII} = \frac{\pi \left(\frac{a}{2}\right)^4}{64}$$

$$J_{zcI} = \frac{2a \left(\frac{a}{2}\right)^3}{12}$$

$$J_{zcII} = \frac{a^4}{12}$$

$$J_{zcIII} = \frac{\pi \left(\frac{a}{2}\right)^4}{64}$$



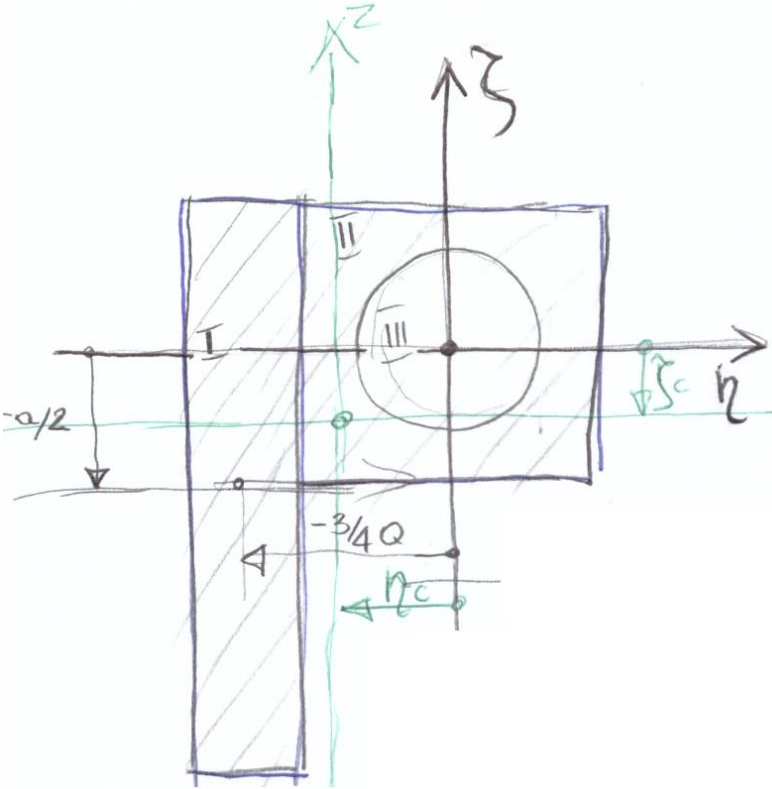
$$\eta_c = \frac{1}{A} \int_A \eta \, dA = \frac{1}{A} [A_I \cdot \eta_{cI} + A_{II} \cdot \eta_{cII} - A_{III} \cdot \eta_{cIII}]$$

$$\eta_c = -\frac{12a}{32-\pi} \approx \boxed{-0.416a}$$

$$\zeta_c = \frac{1}{A} \int_A \zeta \, dA = \frac{1}{A} [A_I \cdot \zeta_{cI} + A_{II} \cdot \zeta_{cII} - A_{III} \cdot \zeta_{cIII}]$$

$$\zeta_c = -\frac{8a}{32-\pi} \approx \boxed{-0.277a}$$

Zadanie 7.2 (c.d.)

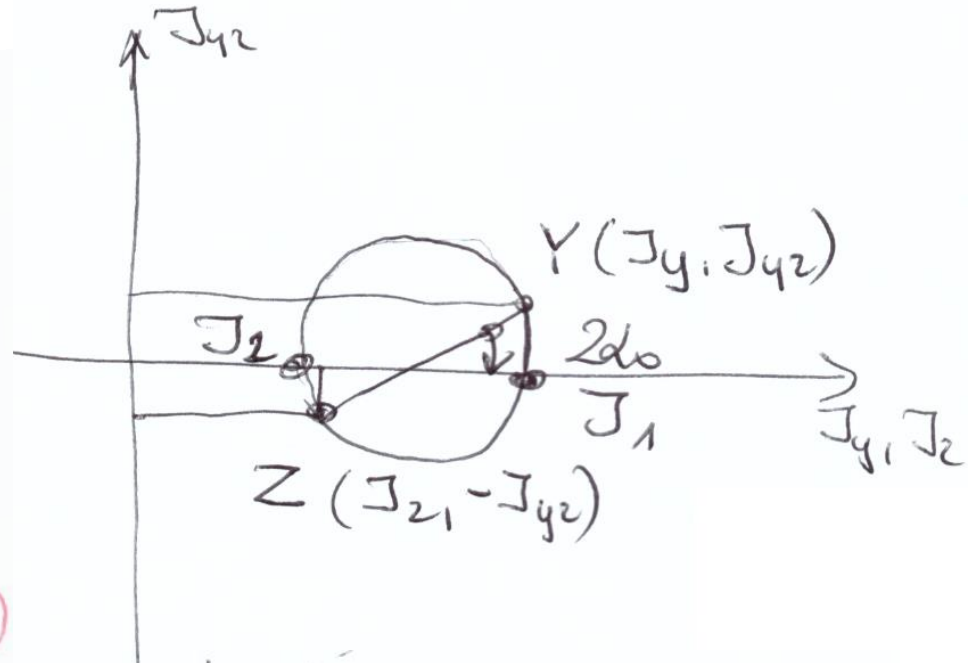
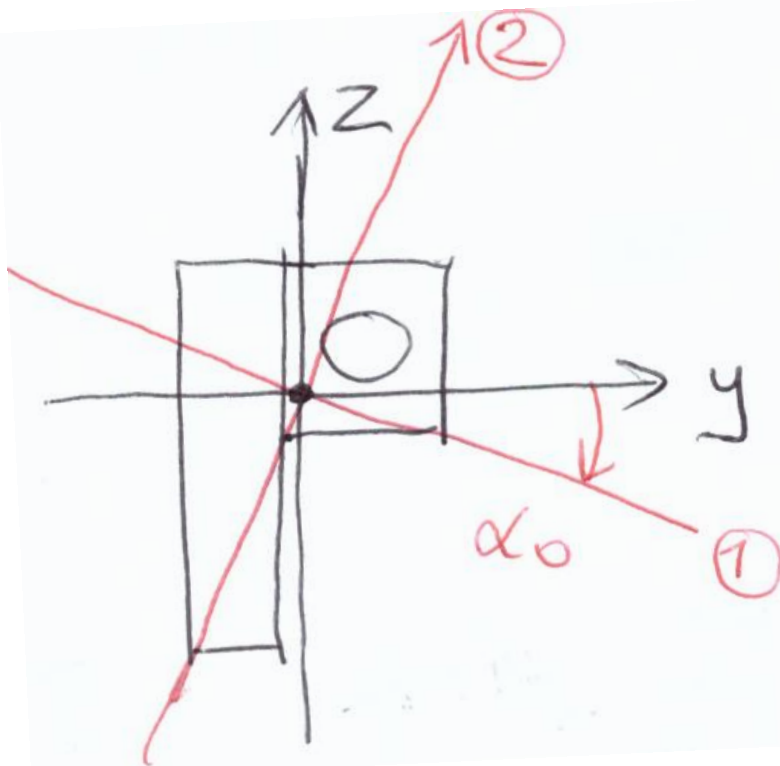


$$\begin{aligned}
 \textcircled{J_y} &= J_{y_{II}} + J_{y_{II}} - J_{y_{III}} = J_{y_{cI}} + A_I (\zeta_{cI}' - \zeta_c')^2 + \\
 &+ J_{y_{cII}} + A_{II} (\zeta_{cII}' - \zeta_c')^2 - J_{y_{cIII}} - A_{III} (\zeta_{cIII}' - \zeta_c')^2 = \\
 &= \frac{a}{2} \frac{(2a)^3}{12} + a^2 \left(-\frac{a}{2} - \left(-\frac{8a}{32-\pi} \right) \right)^2 + \frac{a^4}{12} + a^2 \left(0 - \left(-\frac{8a}{32-\pi} \right) \right)^2 + \\
 &- \frac{\pi \left(\frac{a}{2} \right)^4}{64} - \frac{\pi \left(\frac{a}{2} \right)^2}{4} \left(0 - \left(-\frac{8a}{32-\pi} \right) \right)^2 \approx \boxed{0.525 a^4}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{J_z} &= J_{z_I} + J_{z_{II}} - J_{z_{III}} = J_{z_{cI}} + A_I (\eta_{cI} - \eta_c)^2 + \\
 &+ J_{z_{cII}} + A_{II} (\eta_{cII} - \eta_c)^2 - J_{z_{cIII}} - A_{III} (\eta_{cIII} - \eta_c)^2 = \\
 &= \frac{2a \left(\frac{a}{2} \right)^3}{12} + a^2 \left(-\frac{3}{4}a - \left(-\frac{12a}{32-\pi} \right) \right)^2 + \frac{a^4}{12} + a^2 \left(0 - \left(-\frac{12a}{32-\pi} \right) \right)^2 + \\
 &- \frac{\pi \left(\frac{a}{2} \right)^4}{64} - \frac{\pi \left(\frac{a}{2} \right)^2}{4} \left(0 - \left(-\frac{12a}{32-\pi} \right) \right)^2 \approx \boxed{0.3517 a^4}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{J_{yz}} &= J_{yz_I} + J_{yz_{II}} - J_{yz_{III}} = \\
 &= 0 + A_I (\zeta_{cI}' - \zeta_c') (\eta_{cI} - \eta_c) + 0 + A_{II} (\zeta_{cII}' - \zeta_c') (\eta_{cII} - \eta_c) \\
 &- 0 - A_{III} (\zeta_{cIII}' - \zeta_c') (\eta_{cIII} - \eta_c) = \boxed{0.216 a^4}
 \end{aligned}$$

Zadanie 7.2 (c.d.)



$$J_1 = 0.718 a^4$$

$$J_2 = 0.265 a^4$$

$$\alpha_0 = 36,5^\circ$$